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On intersecting a point set with Euclidean balls [☆]

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Abstract

The *growth function* for a class of subsets \mathcal{C} of a set X is defined by

$$m^{\mathcal{C}}(N) \equiv \max \{ \Delta^{\mathcal{C}}(F) : F \subseteq X, |F| = N \}, \quad N = 1, 2, \dots,$$

where

$$\Delta^{\mathcal{C}}(F) \equiv |\{F \cap C : C \in \mathcal{C}\}|,$$

the number of possible sets obtained by intersecting an element of \mathcal{C} with the set F . Sauer (1972) showed that if \mathcal{C} forms a Vapnik–Chervonenkis class with dimension $V(\mathcal{C})$, then

$$m^{\mathcal{C}}(N) \leq \sum_{j=0}^{V(\mathcal{C})-1} \binom{N}{j} \quad \text{for } N \geq V(\mathcal{C}) - 1.$$

The collection \mathcal{C} of Euclidean balls in R^d has been shown by Dudley (1979) to have VC dimension equal to $d+2$. It is well known, by using a standard geometric transformation, that Sauer's bound gives the exact number of subsets in this case. We give a more direct construction of the subsets picked out by balls, and as a corollary we obtain the number of such subsets.

1. Introduction

Given a class of subsets \mathcal{C} of a set X its *growth function* is defined by

$$m^{\mathcal{C}}(N) \equiv \max \{ \Delta^{\mathcal{C}}(F) : F \subseteq X, |F| = N \}, \quad N = 1, 2, \dots,$$

where

$$\Delta^{\mathcal{C}}(F) \equiv |\{F \cap C : C \in \mathcal{C}\}|$$

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is the number of subsets of F obtained by intersecting F with an element of \mathcal{C} . The class \mathcal{C} is referred to as a Vapnik–Chervonenkis [6] (VC) class if $m^{\mathcal{C}}(N) < 2^N$ for some $N \geq 1$ and in this case its VC dimension is defined by

$$V(\mathcal{C}) \equiv \inf \{N \geq 1: m^{\mathcal{C}}(N) < 2^N\}.$$

Sauer [5] showed that in this case the growth function satisfies

$$m^{\mathcal{C}}(N) \leq \sum_{j=0}^{V(\mathcal{C})-1} \binom{N}{j} \quad \text{for } N \geq V(\mathcal{C}) - 1.$$

Sauer's bound is known to be sharp in some cases. Indeed, this is the case if X is a given infinite set, and \mathcal{C} is taken to be the collection of all subsets of size at most $D - 1$, so that $V(\mathcal{C}) = D$. Interestingly enough, Sauer's bound is also exact when \mathcal{C} consists of all Euclidean balls in R^d , a collection shown by Dudley [1] to have VC dimension equal to $d + 2$. In fact, for any arrangement of N points in general position, Sauer's bound gives the exact number of subsets obtainable by intersecting the point set with Euclidean balls. This fact is known and the well-known proof (see Remark (i)) is to establish a bijection between the collection of subsets picked out by balls and the cells of a certain hyperplane arrangement. Our main result is a more direct construction of the subsets of a point arrangement picked out by balls.

The growth function appears in a fundamental inequality due to Vapnik and Chervonenkis [6], which has seen widespread applications. For example, Haussler and Welzl [3] use this inequality as a starting point in the construction of certain probabilistic algorithms. In such applications, sharper information about the growth function in special cases may contribute to better probabilistic bounds. This result shows that such an improvement is not possible for the class of Euclidean balls.

2. Main result

Given points $x_1, \dots, x_k \in R^d$, where $1 \leq k \leq d + 1$, such that the column $d + 1$ -vectors

$$\begin{bmatrix} 1 \\ x_i \end{bmatrix}, \quad i = 1, \dots, k, \quad \text{are linearly independent,} \quad (\text{I})$$

it is a standard fact that the points possess a unique *circumscribing* sphere (or *circumsphere*) \mathcal{S} , that is, there is a unique sphere of *minimal radius* containing each of them.

A special role is played by the circumspheres for all subsets of size $k = 1, \dots, d + 1$ of a given point set $\{x_1, \dots, x_N\}$. By choosing spheres close to these in a manner described explicitly in Theorem 1, we construct balls which pick out distinct subsets and exhaust all of the possibilities for subsets that can be picked out. First we need to collect some preliminary properties of circumspheres.

The center c and radius r of the circumscribing sphere are referred to as the *circumcenter* and *circumradius* of the points. Note that we allow for the circumradius to be zero since k could be 1. The circumcenter is the unique point in the affine hull of x_1, \dots, x_k which is equidistant from the x_j , $j = 1, \dots, k$, so that there exist unique coefficients $\lambda_1, \dots, \lambda_k$, referred to as the *coefficients of the*

circumcenter such that $\sum_{i=1}^k \lambda_i = 1$ and the point $c \equiv \sum_{i=1}^k \lambda_i x_i$ is equidistant from x_j , $j = 1, \dots, k$. In fact, it is fairly easy to show that when (I) holds, the $k \times k$ matrix

$$M \equiv \begin{bmatrix} 1 & \dots & 1 \\ \langle x_j, x_i - x_1 \rangle \end{bmatrix}_{\substack{i=2,\dots,k \\ j=1,\dots,k}}$$

is invertible and

$$\begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_k \end{bmatrix} = M^{-1} \begin{bmatrix} 1 \\ \frac{1}{2} \{ \|x_2\|^2 - \|x_1\|^2 \} \\ \vdots \\ \frac{1}{2} \{ \|x_k\|^2 - \|x_1\|^2 \} \end{bmatrix}.$$

Note that condition (I) is equivalent to:

$$x_j - x_i, \quad j \neq i, \text{ are linearly independent.} \quad (I')$$

Lemma 1. Given $x_1, \dots, x_k \in R^d$ satisfying (I), the following properties hold:

- (i) if $\lambda_i \neq 0$ then the vectors $x_j - c$, $j \neq i$, are linearly independent, and span a $k - 1$ -dimensional subspace W which is the same for all x_i ,
- (ii) there exists a unique vector $v_i \in W$ which satisfies

$$\langle v_i, x_j - c \rangle = -1 \quad \text{for all } j \neq i,$$

- (iii) the vectors

$$\begin{bmatrix} 1 \\ x_i - c \end{bmatrix}, \quad i = 1, \dots, k,$$

are linearly independent, and

- (iv) if $k > 1$ then $|\{i: \lambda_i > 0\}| > 1$.

From now on we refer to the vectors v_i defined in Lemma 1(ii) as the *dual circumvectors* for the x_i , $i = 1, \dots, k$.

Definition. A collection of points $x_1, \dots, x_k \in R^d$ is in *general position* if $1 \leq k \leq d + 1$, (I) holds, and the coefficients of the circumcenter of the x_i are all nonzero.

It is easy to verify that the general position assumption is valid generically.

Lemma 2. Let $x_1, \dots, x_k \in R^d$ be in general position with circumcenter c , circumradius r , coefficients of the circumcenter λ_i , and dual circumvectors v_1, \dots, v_k . If $\eta_j \geq 0$ with $\sum_{j=1}^k \eta_j = 1$, then the following properties hold:

- (i) $\langle \sum_{j=1}^k \eta_j v_j, x_i - c \rangle = -1 + \eta_i / \lambda_i$, and
- (ii) if $\eta_j > 0$, $j = 1, \dots, k$, there exists $\delta > 0$ such that

$$\left\| c + \varepsilon \sum_{j=1}^k \eta_j v_j - x_i \right\| \begin{cases} < r + \varepsilon & \text{if } \lambda_i > 0, \\ > r + \varepsilon & \text{if } \lambda_i < 0 \end{cases}$$

for all $1 \leq i \leq k$ and $\varepsilon \in (0, \delta)$.

Now we consider arrangements of N points in R^d .

Definition. Points $x_1, \dots, x_N \in R^d$ with $N \geq d + 1$ are in *general position* if for every $J \subseteq \{1, \dots, N\}$ with $1 \leq |J| \leq d + 1$ the points $x_i, i \in J$, are in general position as defined above, and in addition $x_i \notin S$ for all $i \notin J$, where S denotes the circumsphere for $x_i, i \in J$.

Again, it is easy to verify that the general position assumption is valid generically.

Given a sphere S we let S^- denote its *inside* and S^+ denote its *outside*. For a degenerate sphere (radius zero) we take $S^- = \emptyset$.

Theorem 1. Fix points $x_1, \dots, x_N \in R^d$ in general position. For a given set of indices $J = \{p_1, \dots, p_k\}$ with $1 \leq k \leq d + 1$ let S, c, r, λ_i , and v_i denote the corresponding circumsphere, circumcenter, circumradius, coefficients of the circumcenter, and dual circumvectors for $x_{p_i}, i = 1, \dots, k$. Define

$$K(J) = \{i \in J: \lambda_i > 0\} \cup \{i \notin J: x_i \in S^-\}.$$

Then for any $\eta_j > 0$ with $\sum_{j=1}^k \eta_j = 1$ there exists $\delta > 0$ such that for all $0 < \varepsilon < \delta$

$$\left\| c + \varepsilon \sum_{j=1}^k \eta_j v_j - x_i \right\| \begin{cases} < r + \varepsilon & \text{if } i \in K(J), \\ > r + \varepsilon & \text{if } i \notin K(J). \end{cases}$$

Furthermore, if $J_1 \neq J_2$ then $K(J_1) \neq K(J_2)$.

Proof. If $i \in J$ then Lemma 2 gives

$$\left\| c + \varepsilon \sum_{j=1}^k \eta_j v_j - x_i \right\| \begin{cases} < r + \varepsilon & \text{if } \lambda_i > 0, \\ > r + \varepsilon & \text{if } \lambda_i < 0 \end{cases}$$

for $\varepsilon > 0$ sufficiently small. For $i \notin J$ using the general position assumption we have $\|x_i - c\| \neq r$, so since $c + \varepsilon \sum_{j=1}^k \eta_j v_j \rightarrow c$ as $\varepsilon \rightarrow 0$ we have

$$\left\| c + \varepsilon \sum_{j=1}^k \eta_j v_j - x_i \right\| \begin{cases} < r + \varepsilon & \text{if } \|x_i - c\| < r, \\ > r + \varepsilon & \text{if } \|x_i - c\| > r \end{cases}$$

for $\varepsilon > 0$ sufficiently small.

For the second claim, suppose $K(J_1) = K(J_2)$ where $J_1 \neq J_2$ are non-empty index sets whose size is at most $d + 1$. Let c_i, r_i and S_i denote the circumcenter, circumradius and circumsphere corresponding to the subset J_i . There are four cases to consider.

Case 1. If $S_1 = S_2$, without loss of generality suppose $i \in J_2 - J_1$, then $x_i \in S_2 = S_1$ but the fact that $i \notin J_1$ and $x_i \in S_1$ violates the general position assumption.

Case 2. Suppose $S_1^- \subset S_2^-$ (or similarly, suppose $S_2^- \subset S_1^-$). If S_2 is degenerate then J_1 and J_2 each consist of a single index, and the fact that $J_1 \neq J_2$ means the indices are distinct, which leads to a contradiction of the general position assumption. On the other hand, if S_2 is nondegenerate

Lemma 2(iv) guarantees the existence of at least two indices $i \in J_2$ for which the corresponding $\lambda_i > 0$ so $i \in K(J_2)$. At least one of these indices satisfies $x_i \notin S_1^-$, so $i \notin K(J_1)$.

Case 3. If $S_1^- \subset S_2^+$ and $S_2^- \subset S_1^+$ the argument is very similar to the one in Case 2. The spheres might intersect but this intersection can be at most a single point. If both of the spheres are degenerate then either the spheres coincide, in which case we obtain $J_1 = J_2$. If one of the spheres, say S_1 , is nondegenerate, by Lemma 2(iv) there exists $i \in J_1$ such that the corresponding $\lambda_i > 0$ and $x_i \notin S_2$. Thus, $i \in K(J_1)$. On the other hand, $x_i \in S_2^+$ so that $i \notin K(J_2)$, a contradiction.

Case 4. If none of the above cases occur, then $S_1 \neq S_2$ and $S_1 \cap S_2$ is neither empty nor a single point. It follows that $c_1 \neq c_2$ and the $S_1 \cap S_2$ forms a $(d-1)$ -sphere which is contained in some hyperplane H . By making a change of coordinates, we may assume $c_1 = A_1 e_1$ and $c_2 = A_2 e_1$ where $e_1 \equiv (1, 0, \dots, 0)^T$ and $A_1 < A_2$. It follows that $H = \{x \in R^d: \langle x, e_1 \rangle = A\}$ for some $A \in R$. Now either $A_1 < A$ or $A < A_2$ (or possibly both), and the argument is the same in either case, so assume $A_1 < A$. Define half-spaces

$$H^+ \equiv \{x \in R^d: \langle x, e_1 \rangle \geq A\} \quad \text{and} \quad H^- \equiv \{x \in R^d: \langle x, e_1 \rangle \leq A\}.$$

It follows that $S_1 \cap (S_2 \cup S_2^+) \subseteq H^-$ and $S_1 \cap (S_2 \cup S_2^-) \subseteq H^+$.

If $i \in J_1$ and the corresponding $\lambda_i > 0$ then $x_i \in S_1$ and $i \in K(J_1)$. Thus, $i \in K(J_2)$ which gives $x_i \in S_2 \cup S_2^-$. But then $x_i \in S_1 \cap (S_2 \cup S_2^-)$ so $x_i \in H^+$. Similarly, if $i \in J_1$ and $\lambda_i < 0$ then $x_i \in S_1$ and $i \notin K(J_1)$. Thus, $i \notin K(J_2)$ which gives $x_i \in S_2 \cup S_2^+$. But then $x_i \in S_1 \cap (S_2 \cup S_2^+)$ so $x_i \in H^-$. Thus, we have for $i \in J_1$, $\lambda_i > 0 \Rightarrow \langle x_i, e_1 \rangle \geq A$ and $\lambda_i < 0 \Rightarrow \langle x_i, e_1 \rangle \leq A$. These inequalities give

$$\begin{aligned} 0 &= \left\langle \sum_{i \in J_1} \lambda_i (x_i - c_1), e_1 \right\rangle = \sum_{\lambda_i > 0} \lambda_i \langle x_i, e_1 \rangle + \sum_{\lambda_i < 0} \lambda_i \langle x_i, e_1 \rangle - \langle c_1, e_1 \rangle \\ &\geq \sum_{\lambda_i > 0} \lambda_i A + \sum_{\lambda_i < 0} \lambda_i A - \langle c_1, e_1 \rangle = \sum_{i \in J_1} \lambda_i A - \langle c_1, e_1 \rangle = A - \langle c_1, e_1 \rangle = A - A_1, \end{aligned}$$

which is a contradiction. \square

Theorem 1 gives centers and radii for balls which cover distinct *non-empty* subsets $\{x_i, i \in K(J)\}$ of a given point set $\{x_1, \dots, x_N\}$.

Corollary 1. For any $x_1, \dots, x_N \in R^d$ with $N \geq d+1$ in general position the number of subsets of the form $B(x, r) \cap \{x_i, i = 1, \dots, N\}$ is given by

$$\sum_{j=0}^{d+1} \binom{N}{j}.$$

Proof. By the theorem, each of the sets $\{x_i, i \in K(J)\}$ may be obtained by intersecting a ball with $\{x_i, i = 1, \dots, N\}$. Each $K(J)$ is non-empty by Lemma 1(iv), so we can also define $K(\emptyset) = \emptyset$, and the $K(J)$ for $J \subseteq \{1, \dots, N\}$ with $|J| \leq d+1$ remain distinct. The fact that the mapping $J \rightarrow K(J)$ is one-to-one gives

$$\sum_{j=0}^{d+1} \binom{N}{j}$$

as a lower bound for the number of subsets. Since this is the same as the upper bound given by Sauer's [5] Lemma, we obtain the result. \square

3. Remarks

(i) Theorem 1 provides a direct construction of the sets obtained by intersecting a point set with balls. The following alternative construction is a well-known (though less direct) method for giving the point sets picked out. It is based on a duality property of the geometric transformation, discussed in [2, Section 1.4], taking a point $(u, v) \in R^d \times R$ to the hyperplane

$$H_{(u,v)} = \{(x, y) \in R^d \times R: y = 2\langle u, x \rangle - \|v\|^2\}.$$

Let $H_{(u,v)}^+$ ($H_{(u,v)}^-$) denote the points lying above (below) this hyperplane.

Let x_1, \dots, x_N be in general position in R^d and let $U = \{(x, y) \in R^d \times R: y = \|x\|^2\}$ denote the unit paraboloid. Lift each point x_i to $(x_i, \|x_i\|^2) \in U$, and let H_i denote the supporting hyperplane of U at this point, so that $H_i = H_{(x_i, \|x_i\|^2)}$. The arrangement $\{H_i, i = 1, \dots, n\}$ defines

$$M := \sum_{j=0}^{d+1} \binom{N}{j}$$

non-empty cells, each of which corresponds to a distinct index set J with the property that for a point (x^*, y^*) in the cell we have

$$(x^*, y^*) \in \begin{cases} H_i^- & \text{if } i \in J, \\ H_i^+ & \text{if } i \notin J. \end{cases}$$

From duality it follows that

$$(x_i, y_i) \in \begin{cases} H_{(x^*, y^*)}^- & \text{if } i \in J, \\ H_{(x^*, y^*)}^+ & \text{if } i \notin J. \end{cases}$$

As a consequence, the ball obtained by projecting $H_{(x^*, y^*)}^- \cap U$ to R^d picks out the points $x_i, i \in J$.

(ii) Naiman and Wynn [4] give examples of indicator function identities of the form

$$\sum_{J \subseteq \{1, \dots, N\}} c_J I\left(\bigcap_{i \in J} B(x_i, r)\right) = 0,$$

which are valid for all $r \geq 0$, for given points x_1, \dots, x_N . It is possible to use Theorem 1 to determine all of the possible identities of this form when x_1, \dots, x_N are in general position. In fact, then the above identity is valid for all $r \geq 0$ if and only if $\sum_{J: J \subseteq K} c_J = 0$ whenever K is one of the index sets picked out by a ball.

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Appendix A. Proofs of the lemmas

Proof of Lemma 1. If $\sum_{j \neq i} \mu_j (x_j - c) = 0$ we consider two cases. First, if $\sum_{j \neq i} \mu_j \neq 0$ then $c = \sum_{j \neq i} \mu_j x_j / \sum_{j \neq i} \mu_j$, which means λ_i must be zero, a contradiction. On the other hand, if $\sum_{j \neq i} \mu_j = 0$ then $0 = \sum_{j \neq i} \mu_j (x_j - c) = \sum_{j \neq i} \mu_j (x_j - x_i)$. Using (I') we see that $\mu_j = 0$ for $j \neq i$ so all of the μ_i vanish. This proves the first claim of (i). The second follows from the fact that $\sum_{j=1}^k \lambda_j (x_j - c) = 0$.

(ii) follows from the fact that the $x_j - c$, $j \neq i$, form a basis for W .

For (iii) suppose

$$\sum_{i=1}^k \mu_i \begin{bmatrix} 1 \\ x_i - c \end{bmatrix} = 0.$$

It follows that $\sum_{i=2}^k \mu_i (x_i - x_1) = 0$ and by (I') we obtain $\mu_i = 0$ for $i = 1, \dots, k$.

Finally, to prove (iv) suppose, without loss of generality, that $\lambda_1 > 0$ and $\lambda_2, \dots, \lambda_k \leq 0$. Then, letting r denote $\|x_i - c\|$ we have

$$0 = \left\langle \sum_{i=1}^k \lambda_i (x_i - c), x_1 - c \right\rangle = \lambda_1 r^2 + \sum_{i=2}^k \lambda_i \langle x_i - c, x_1 - c \rangle \geq \sum_{i=1}^k \lambda_i r^2$$

by the Cauchy–Schwartz inequality. Since $\sum_{i=1}^k \lambda_i = 1$ this is a contradiction when $k > 1$. \square

Proof of Lemma 2. Since $\sum_{j=1}^k \lambda_j (x_j - c) = 0$ we see that

$$\begin{aligned} 0 &= \left\langle v_i, \sum_{j=1}^k \lambda_j (x_j - c) \right\rangle = \sum_{j \neq i} \lambda_j \langle v_i, x_j - c \rangle + \lambda_i \langle v_i, x_i - c \rangle \\ &= - \sum_{j \neq i} \lambda_j + \lambda_i \langle v_i, x_i - c \rangle = \lambda_i - 1 + \lambda_i \langle v_i, x_i - c \rangle. \end{aligned}$$

This gives $\langle v_i, x_i - c \rangle = 1/\lambda_i - 1$. Since

$$\left\langle \sum_{j=1}^k \eta_j v_j, x_i - c \right\rangle = - \sum_{j \neq i} \eta_j + \eta_i \langle v_i, x_i - c \rangle$$

(i) follows.

For (ii) define a $d \times (k-1)$ matrix

$$P_i \equiv [x_1 - c, \dots, x_{i-1} - c, x_{i+1} - c, \dots, x_k - c]$$

so that $v_i = -(P_i^T)^{-1} \mathbf{1}$. It follows that there exists a constant $C > 0$ such that $\sup_{1 \leq i \leq k} \|v_i\|^2 \leq C$. If $\lambda_i > 0$, using Lemma 1 we see that

$$\left\| c + \varepsilon \sum_{j=1}^k \eta_j v_j - x_i \right\|^2 - (r + \varepsilon)^2 = \left\| \varepsilon \sum_{j=1}^k \eta_j v_j - (x_i - c) \right\|^2 - (r + \varepsilon)^2$$

$$= \varepsilon^2 \left\| \sum_{j=1}^k \eta_j v_j \right\|^2 - \varepsilon^2 - 2 \frac{\eta_i}{\lambda_i} \varepsilon \leq C \varepsilon^2 - 2 \frac{\eta_i}{\lambda_i} \varepsilon < 0$$

provided $0 < \varepsilon < 2\eta_i/(C\lambda_i)$. If $\lambda_i < 0$ then we obtain

$$\left\| c + \varepsilon \sum_{j=1}^k \eta_j v_j - x_i \right\|^2 - (r + \varepsilon)^2 \geq -\varepsilon^2 - \frac{2\eta_i}{\lambda_i} \varepsilon > 0$$

provided $0 < \varepsilon < -2\eta_i/\lambda_i$. We see that the statement of the lemma holds for δ sufficiently small. \square

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